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Holographic superconductors with Lifshitz scaling

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Abstract

Black holes in asymptotically Lifshitz spacetime provide a window onto finite temperature effects in strongly coupled Lifshitz models. We add a Maxwell gauge field and charged matter to a recently proposed gravity dual of (2+1)-dimensional Lifshitz theory. This gives rise to charged black holes with scalar hair, which correspond to the superconducting phase of holographic superconductors with $z > 1$ Lifshitz scaling. Along the way we analyze the global geometry of static, asymptotically Lifshitz black holes at the arbitrary critical exponent $z > 1$. In all known exact solutions there is a null curvature singularity in the black hole region, and, by a general argument, the same applies to generic Lifshitz black holes.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

A holographic superconductor [1, 2] is dual to an asymptotically AdS spacetime with a charged black hole carrying scalar hair. It is a theoretical construction where a superconducting phase transition in a strongly coupled system is studied via the AdS/CFT correspondence [3]. Having a black hole places the system at finite temperature, the black hole charge gives rise to a chemical potential in the dual theory, and the scalar field hair signals the condensation of a charged operator in the dual theory. Without a chemical potential all temperatures are equivalent due to the underlying conformal symmetry and there can be no phase transition. With a chemical potential, on the other hand, a new scale is introduced that allows for a phase transition at some critical temperature.

The black holes in question are found as solutions of AdS gravity coupled to a Maxwell gauge field and matter in the form of a charged scalar. Black holes that are neutral under the

Maxwell field do not develop any hair. Near a charged black hole, however, the gauge field sourced by the black hole couples to the charged scalar and induces a negative mass squared sufficient for condensation provided the temperature is low enough. It was observed in [4] that even a neutral scalar can lead to condensation below a non-zero critical temperature due to the existence of a new and effectively lower Breitenlohner–Freedman bound [5] near the horizon of a low temperature, near extremal, AdS–Reissner–Nordström black hole. See [6] and [7] for recent reviews of holographic superconductivity.

In the present paper we show that similar phenomena can occur in a theory exhibiting Lifshitz scaling:

$$t \rightarrow \lambda^z t, \quad \mathbf{x} \rightarrow \lambda \mathbf{x}, \quad (1.1)$$

with $z \neq 1$. Models with scaling of this type have, for example, been used to model the quantum critical behavior in strongly correlated electron systems [8–11]. Our starting point is the model proposed by Kachru *et al* [12] for the holographic study of strongly coupled (2+1)-dimensional systems with Lifshitz scaling. While this model in itself does not support a phase transition to a superconductor it turns out that a relatively simple extension does.

A quantum critical point exhibiting dynamical scaling of the form (1.1) has a gravitational dual description in terms of a spacetime metric of the form

$$ds^2 = L^2 \left(-r^{2z} dt^2 + \frac{dr^2}{r^2} + r^2 d^2\mathbf{x} \right), \quad (1.2)$$

which is invariant under the transformation

$$t \rightarrow \lambda^z t, \quad r \rightarrow \frac{r}{\lambda}, \quad \mathbf{x} \rightarrow \lambda \mathbf{x}. \quad (1.3)$$

Here L is a characteristic length scale and the coordinates (t, r, x^1, x^2) are taken to be dimensionless. Critical points with $z > 1$ are often said to be non-relativistic. This can be motivated by considering null geodesics at fixed $r = r_0$ in the Lifshitz background (1.2). Along such a geodesic one finds that $(d\mathbf{x}/dt)^2 = r_0^{2z-2}$ and so for $z > 1$ the effective speed of light in the boundary theory diverges as $r_0 \rightarrow \infty$.

In [12] it was shown how the fixed point geometry (1.2), with $z > 1$, can be obtained from an action coupling (3+1)-dimensional gravity with a negative cosmological constant to Abelian two- and three-form field strengths:

$$S = \int d^4x \sqrt{-g} (R - 2\Lambda) - \frac{1}{2} \int *F_{(2)} \wedge F_{(2)} - \frac{1}{2} \int *H_{(3)} \wedge H_{(3)} - c \int B_{(2)} \wedge F_{(2)}, \quad (1.4)$$

with $H_{(3)} = dB_{(2)}$ and the cosmological constant and the coupling between the form fields given by $\Lambda = -\frac{z^2+z+4}{2L^2}$ and $c = \frac{\sqrt{2z}}{L}$. The equations of motion for the form fields can be written as

$$d * F_{(2)} = -c H_{(3)}, \quad d * H_{(3)} = -c F_{(2)}, \quad (1.5)$$

and the Einstein equations are

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{1}{2} (F_{\mu\lambda} F_{\nu}^{\lambda} - \frac{1}{4} g_{\mu\nu} F_{\lambda\sigma} F^{\lambda\sigma}) + \frac{1}{4} (H_{\mu\lambda\sigma} H_{\nu}^{\lambda\sigma} - \frac{1}{6} g_{\mu\nu} H_{\lambda\sigma\rho} H^{\lambda\sigma\rho}). \quad (1.6)$$

Black holes in this (3+1)-dimensional gravity theory were considered in [13], where numerical black hole solutions were found at $z = 2$ and used to study finite temperature effects in the dual (2+1)-dimensional system. Related work on Lifshitz black holes can be found in [14], where topological black holes with hyperbolic horizons were included, and in

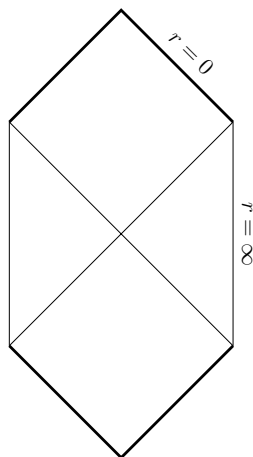


Figure 1. Conformal diagram for a Lifshitz black hole. The thick lines denote null curvature singularities.

[15, 16], where black holes at general values of z were considered⁴. These Lifshitz black holes carry a charge that couples to the two-form field strength $F_{(2)}$, but they are in many respects more analogous to AdS–Schwarzschild black holes than charged AdS–Reissner–Nordström black holes. In particular, since the black hole charge cannot be varied independently of the black hole area, there is only a one-parameter family of black hole solutions for a given value of z .

We will see below that without additional ingredients the underlying Lifshitz symmetry prevents the system from undergoing phase transitions. We therefore extend the system to include a Maxwell field, with field strength $\mathcal{F}_{(2)}$, and a scalar field ψ that is charged under the new gauge field but neutral under the original Lifshitz fields $F_{(2)}$ and $H_{(3)}$. This turns out to be sufficient in order to observe a superconducting phase transition characterized by Lifshitz scaling with $z > 1$. In our modified theory, it is the Maxwell field $\mathcal{F}_{(2)}$ that corresponds to physical electromagnetism while the original Lifshitz gauge fields are viewed as an auxiliary construction, whose only role is to modify the asymptotic symmetry of the geometry from AdS to Lifshitz.

The plan of the rest of the paper is as follows. We begin in section 2 with a brief review of Lifshitz black holes in the model (1.4). We extend previous treatments by considering the global geometry, including the black hole interior. In those cases where an exact solution is known, we find a null curvature singularity at $r = 0$ and a Carter–Penrose diagram as shown in figure 1. We show that a null singularity is generic for black holes in this model. In section 3 we generalize the model to include an additional Maxwell field. We exhibit a family of exact solutions at $z = 4$, which describe Lifshitz black holes that are charged under the new Maxwell field and obtain numerical solutions for charged black holes at other values of $z > 1$. These black holes can be viewed as the Lifshitz analog of AdS–Reissner–Nordström black holes. In section 4 we further extend the model by adding a charged scalar field and look for the black hole instability that signals the onset of superconductivity. Finally, we conclude with some final remarks in section 5.

⁴ Black hole solutions in other gravity models exhibiting Lifshitz scaling have been considered in [17–19].

2. Lifshitz black holes revisited

The action (1.4) is known to have spherically symmetric static black hole solutions of the form

$$ds^2 = L^2 \left(-r^{2z} f(r)^2 dt^2 + \frac{g(r)^2}{r^2} dr^2 + r^2 (d\theta^2 + \chi(\theta)^2 d\varphi^2) \right), \quad (2.1)$$

with

$$\chi(\theta) = \begin{cases} \sin \theta & \text{if } k = 1, \\ \theta & \text{if } k = 0, \\ \sinh \theta & \text{if } k = -1, \end{cases}$$

where $k = +1, 0, -1$ corresponds to a spherical, flat and hyperbolic horizon, respectively. An asymptotically Lifshitz black hole with a non-degenerate horizon has $f(r), g(r) \rightarrow 1$ as $r \rightarrow \infty$ and a simple zero of both $f(r)^2$ and $g(r)^{-2}$ at the horizon $r = r_0$.

2.1. Exact solutions and global extensions

Several families of numerical solutions of this general form have been found [13–15] along with a couple of isolated exact black hole solutions. These are a $z = 2$ topological black hole with a hyperbolic horizon [14]:

$$f(r) = \frac{1}{g(r)} = \sqrt{1 - \frac{1}{2r^2}}, \quad (2.2)$$

and a $z = 4$ black hole [15]:

$$f(r) = \frac{1}{g(r)} = \sqrt{1 + \frac{k}{10r^2} - \frac{3k^2}{400r^4}}, \quad (2.3)$$

with $k = \pm 1$. The $k = +1$ case has a spherical horizon while $k = -1$ corresponds to a topological black hole with a hyperbolic horizon. When we extend the system to include an additional Maxwell gauge field (see section 3 below) we will see that the exact $z = 4$ solution (2.3) is a special case of a one-parameter family of exact solutions in the extended theory.

We start our discussion by working out the global geometry of the exact $z = 2$ topological black hole (2.2) and constructing the associated conformal diagram. In order to study the interior geometry of a black hole, one looks for coordinates that are non-singular at the black hole horizon and allow the solution to be extended into the black hole. The first step is to transform to a tortoise coordinate r_* for which

$$ds^2 = L^2 \left(r^4 \left(1 - \frac{1}{2r^2} \right) (-dt^2 + dr_*^2) + r^2 (d\chi^2 + \sinh^2 \chi d\varphi^2) \right). \quad (2.4)$$

Here r is to be viewed as a function of r_* determined via

$$\frac{dr}{dr_*} = r^3 \left(1 - \frac{1}{2r^2} \right), \quad (2.5)$$

which integrates to

$$r = \frac{1}{\sqrt{2} \sqrt{1 - \exp(r_* - r_*^\infty)}}. \quad (2.6)$$

As usual, the tortoise coordinate r_* goes to $-\infty$ at the horizon but note that it goes to a finite value in the $r \rightarrow \infty$ asymptotic region. This is easily seen to be a generic feature of asymptotically Lifshitz black holes for any $z > 1$.

Next we form the null combinations $v = t + r_*$, $u = t - r_*$ and perform a conformal reparametrization

$$\begin{aligned} v &\rightarrow V = \exp\left[\frac{1}{2}(v - r_*^\infty)\right], \\ u &\rightarrow U = -\exp\left[-\frac{1}{2}(u + r_*^\infty)\right], \end{aligned} \tag{2.7}$$

which brings the metric into the form

$$ds^2 = L^2(-4r^4 dU dV + r^2(d\theta^2 + \sinh^2 \theta d\varphi^2)). \tag{2.8}$$

In these coordinates the geometry is manifestly nonsingular at the event horizon and the solution can be extended to $r < 1/\sqrt{2}$.

In general, when extending a black hole solution through a non-degenerate horizon, the required conformal reparametrization is of the form

$$\begin{aligned} v &\rightarrow V = \alpha \exp(\kappa v), \\ u &\rightarrow U = -\alpha \exp(-\kappa u), \end{aligned} \tag{2.9}$$

where α is an arbitrary constant and κ is the surface gravity of the black hole in question. The surface gravity determines the Hawking temperature, $T_H = \frac{\kappa}{2\pi}$, and it is easily checked by other means that $T_H = \frac{1}{4\pi}$ is the correct value for the $z = 2$ black hole in (2.2). We have chosen the value of the constant α in (2.7) such that $UV \rightarrow -1$ in the $r \rightarrow \infty$ asymptotic limit.

The $z = 2$ topological black hole has a curvature singularity at $r = 0$, which can for example be seen by computing the Ricci scalar⁵

$$R = \frac{1}{r^2 L^2} (1 - 22 r^2). \tag{2.10}$$

It follows from

$$-UV = 1 - \frac{1}{2r^2} \tag{2.11}$$

that $r \rightarrow \infty$ corresponds to $UV \rightarrow -1$ and $r \rightarrow 0$ to $UV \rightarrow \infty$. The Carter–Penrose conformal diagram⁶ in figure 1 is then obtained by writing

$$V = \tan \frac{\pi P}{2}, \quad U = \tan \frac{\pi Q}{2}. \tag{2.12}$$

The $z = 4$ case can be dealt within a similar manner. There is a null curvature singularity at $r = 0$ and the tortoise coordinate for a $z = 4$ black hole is given by

$$r_* - r_*^\infty = \frac{1}{2b_1(b_1 + b_2)} \log\left(1 - \frac{b_1}{r^2}\right) + \frac{1}{2b_2(b_1 + b_2)} \log\left(1 + \frac{b_2}{r^2}\right), \tag{2.13}$$

with $b_1 = \frac{1}{20}(2|k| - k)$ and $b_2 = \frac{1}{20}(2|k| + k)$. The change of coordinates to

$$V = \exp[b_1(b_1 + b_2)(v - r_*^\infty)], \quad U = -\exp[-b_1(b_1 + b_2)(u + r_*^\infty)] \tag{2.14}$$

renders the metric nonsingular at the horizon:

$$ds^2 = L^2 \left(-\frac{r^8}{\kappa^2} \left(1 + \frac{b_2}{r^2}\right)^{1 - \frac{b_1}{b_2}} dU dV + r^2(d\theta^2 + \chi^2(\theta) d\varphi^2) \right). \tag{2.15}$$

⁵ The fixed point geometry (1.2) is also singular at $r = 0$ but in a relatively mild way. All scalar invariants constructed from the Riemann tensor are finite but tidal effects between neighboring null geodesics diverge as $r \rightarrow 0$. See for example [6].

⁶ We refer to figure 1 as a conformal diagram but it should be kept in mind that the boundary at $r \rightarrow \infty$ is in fact not conformally flat due to the asymmetric scaling of the spatial and time directions in asymptotically Lifshitz spacetime.

The Hawking temperature of these $z = 4$ black holes is

$$T_H = \frac{1}{200\pi}(2 - k), \tag{2.16}$$

and the Ricci scalar is given by

$$R = \frac{1}{L^2} \left(\frac{9k^2}{200r^4} - \frac{3k}{5r^2} - 54 \right). \tag{2.17}$$

There is a null curvature singularity at $r \rightarrow 0$ and the conformal diagram is that of figure 1. In section 2.3 below, we present a general argument that this conformal diagram applies to all black hole solutions of the system of equations (1.5)–(1.6).

2.2. Lifshitz black holes at general $z > 1$

The exact solutions offer a glimpse at the parameter space of Lifshitz black holes at isolated points. More generic solutions can be obtained numerically. We are interested in the global geometry including the black hole interior. For this we take our cue from the extension of the exact solution described above and write the metric in the form

$$ds^2 = L^2[-e^{2\rho(v,u)} dv du + e^{-2\phi(v,u)}(d\theta^2 + \chi(\theta)^2 d\varphi^2)]. \tag{2.18}$$

The notation is descended from two-dimensional gravity and allows the field equations to be written in a relatively economical way. With this ansatz the metric is characterized by two field variables, ρ and ϕ , which determine the local scale of the v, u plane and the scale of the transverse two-manifold, respectively. The null coordinates v, u are related to the t, r coordinates in (2.1) via $v = t + r_*$, $u = t - r_*$, with the tortoise coordinate r_* given by

$$r_* - r_*^\infty = - \int_r^\infty \frac{d\tilde{r}}{\tilde{r}^{z+1}} \frac{g(\tilde{r})}{f(\tilde{r})}. \tag{2.19}$$

The Lifshitz fixed point geometry (1.2) has $f(r) = g(r) = 1$ and in that case

$$r_* = r_*^\infty - \frac{1}{z r^z}. \tag{2.20}$$

More generally, we see from (2.19) that r_* goes to a finite value asymptotically for any Lifshitz spacetime, for which $f(r), g(r) \rightarrow 1$ as $r \rightarrow \infty$.

2.2.1. The equations of motion. With the following ansatz for the form fields:

$$F_{(2)} = L f_{vu}(v, u) dv \wedge du, \tag{2.21}$$

$$H_{(3)} = L^2(h_v(v, u) dv - h_u(v, u) du) \wedge d\theta \wedge \chi(\theta) d\varphi, \tag{2.22}$$

their field equations (1.5) become

$$-\partial_v(e^{-2\rho-2\phi} f_{vu}) = \sqrt{\frac{z}{2}} h_v, \tag{2.23}$$

$$\partial_u(e^{-2\rho-2\phi} f_{vu}) = \sqrt{\frac{z}{2}} h_u, \tag{2.24}$$

$$-\partial_v(e^{2\phi} h_u) + \partial_u(e^{2\phi} h_v) = \sqrt{2z} f_{vu}. \tag{2.25}$$

These can in turn be re-expressed as a single second-order equation,

$$\partial_v \partial_u \tilde{f} + \partial_v \phi \partial_u \tilde{f} + \partial_u \phi \partial_v \tilde{f} + \frac{z}{2} e^{2\rho} \tilde{f} = 0, \tag{2.26}$$

where we have defined

$$\tilde{f} \equiv e^{-2\rho-2\phi} f_{vu}. \quad (2.27)$$

The equations of motion for ϕ and ρ are obtained from (1.6):

$$-\partial_v \partial_u \phi + 2\partial_v \phi \partial_u \phi + \frac{k}{4} e^{2\rho+2\phi} + \frac{(z^2+z+4)}{8} e^{2\rho} - \frac{1}{4} \tilde{f}^2 e^{2\rho+4\phi} = 0, \quad (2.28)$$

$$-\partial_v \partial_u \rho + \partial_v \phi \partial_u \phi + \frac{k}{4} e^{2\rho+2\phi} - \frac{1}{2} \tilde{f}^2 e^{2\rho+4\phi} + \frac{1}{2z} e^{4\phi} \partial_v \tilde{f} \partial_u \tilde{f} = 0. \quad (2.29)$$

The conformal reparametrization (2.9) will render the metric nondegenerate at the horizon. The form of the field equations remains the same in the V, U coordinate system with a transformed conformal factor:

$$e^{2\rho(v,u)} = (-\kappa^2 UV) e^{2\rho(V,U)}. \quad (2.30)$$

We now adapt a simple numerical method, which was originally developed for the study of two-dimensional black holes [20], to the case at hand. We introduce a new spatial variable

$$s \equiv -UV \quad (2.31)$$

and restrict our attention to static configurations. The field equations become

$$0 = s \tilde{f}'' + \tilde{f}' + 2s\phi' \tilde{f}' - \frac{z}{2} \tilde{f} e^{2\rho}, \quad (2.32)$$

$$0 = s\phi'' + \phi' - 2s\phi'^2 + \frac{k}{4} e^{2\rho+2\phi} + \frac{(z^2+z+4)}{8} e^{2\rho} - \frac{1}{4} \tilde{f}^2 e^{2\rho+4\phi}, \quad (2.33)$$

$$0 = s\rho'' + \rho' - s\phi'^2 + \frac{k}{4} e^{2\rho+2\phi} - \frac{1}{2} \tilde{f}^2 e^{2\rho+4\phi} - \frac{s}{2z} e^{4\phi} \tilde{f}'^2, \quad (2.34)$$

where primes denote derivatives with respect to s .

The Lifshitz geometry (1.2) is given by

$$\tilde{f} = \sqrt{\frac{z(z-1)}{2}} r^2, \quad e^{-\phi} = r, \quad \kappa^2 s e^{2\rho(s)} = r^{2z}, \quad (2.35)$$

with $r(s)$ is obtained from (2.20) and (2.31). The horizon at $r = 0$ is singular and as a result κ can take any value in the Lifshitz geometry. The $z = 2$ topological black hole has $\tilde{f} = r^2$ with $r(s) = 1/\sqrt{2(1-s)}$, while the $z = 4$ black holes turn out to have $\tilde{f} = \sqrt{6}(r^2 + \frac{k}{20})$, with $r(s)$ obtained via $\frac{dr}{dr_s} = r^5(1 + \frac{k}{10r^2} - \frac{3}{400r^4})$ and (2.31).

2.2.2. Numerical solutions. The event horizon is at $s = 0$ and s is negative inside the black hole. With the assumption of a regular horizon we can read off the following relations among initial values at $s = 0$:

$$\phi'(0) = \frac{1}{4} \left(-k e^{2\phi(0)} - \frac{(z^2+z+4)}{2} + \tilde{f}(0)^2 e^{4\phi(0)} \right) e^{2\rho(0)}, \quad (2.36)$$

$$\tilde{f}'(0) = \frac{z}{2} \tilde{f}(0) e^{2\rho(0)}, \quad (2.37)$$

$$\rho'(0) = \frac{1}{2} \left(-\frac{k}{2} e^{2\phi(0)} + \tilde{f}(0)^2 e^{4\phi(0)} \right) e^{2\rho(0)}. \quad (2.38)$$

These initial values can now be used to start a numerical integration of the system (2.32)–(2.34) from near $s = 0$, either outward toward $s > 0$ or into the black hole interior at $s < 0$. Inequivalent solutions are parametrized by $\phi(0)$ and $\tilde{f}(0)$. The $\phi(0)$ initial value gives the value of the area coordinate at the horizon through the relation $r = e^{-\phi}$ and $\tilde{f}(0)$ determines the magnitude of the radial two-form field at the horizon. A shift of $\rho(0)$ amounts to a global rescaling of the s coordinate and does not affect the geometry.

As discussed in [13], $\phi(0)$ and $\tilde{f}(0)$ cannot be varied independently while preserving the asymptotically Lifshitz character of the geometry. In the $z = 2$ case considered in that paper, there is a zero mode of the linearized system of equations near the Lifshitz fixed point, which must be set to zero for the system to approach the fixed point geometry (1.2) as $r \rightarrow \infty$. This requires a fine-tuning of parameters which uniquely determines $\tilde{f}(0)$ in terms of $\phi(0)$ or vice versa. In [15], it was shown that at $z > 2$, the zero mode is replaced by a power-law growing mode which must be set to zero to obtain an asymptotically Lifshitz geometry. The question was more subtle at $z < 2$ where instead there is a mode with a weak power-law fall-off but a finite energy argument [15, 21] may be employed to conclude that this mode must also be set to zero⁷.

With the variables that we are using here there is a simple criterion to identify the subset of initial values $\phi(0)$ and $\tilde{f}(0)$ that led to asymptotically Lifshitz black holes for any value of $z > 1$. We already learned from (2.19) that the numerical integration toward $s > 0$, i.e. toward the asymptotic region, will terminate at a finite value $s = s_\infty$. We also know that $e^{2\rho(v,u)} \rightarrow r^{2z}$, as $r \rightarrow \infty$ for asymptotically Lifshitz geometry. For a given $\phi(0)$ we tune $\tilde{f}(0)$ until the combination $\rho(s) + z\phi(s)$ goes to a finite value in the limit $s \rightarrow s_\infty$. Furthermore, through the transformation rule (2.30) for the conformal factor we can read off the Hawking temperature of the black hole:

$$T_H = \frac{\kappa}{2\pi} = \frac{1}{2\pi\sqrt{s_\infty}} \lim_{s \rightarrow s_\infty} e^{-\rho(s) - z\phi(s)}, \quad (2.39)$$

once we have fine-tuned the initial data to give a finite limit. This method works the same way for both $z > 2$ and $z < 2$. Although the unwanted mode is slowly decaying as a function of r when $r \rightarrow \infty$ in the $z < 2$ case, it has a growing amplitude as a function of s due to the rapid growth of $\frac{dr}{ds}$ as $s \rightarrow s_\infty$.

A numerical solution for both the exterior and interior geometry can now be obtained by separately integrating toward positive and negative s , using the same fine-tuned initial data, and then patching the two solutions together across $s = 0$. Figure 2 shows a numerical black hole solution at $z = 4$ obtained in this manner. The curvature singularity is at $s \rightarrow -\infty$ in these coordinates and the numerical evaluation will break down before it is reached.

2.3. Global geometry of generic Lifshitz black holes

We now give an argument that the conformal diagram in figure 1 applies to a generic Lifshitz black hole solution that has a single non-degenerate horizon and is obtained from the action (1.4), for some value of $z > 1$. To do that, one examines the asymptotic behavior of solutions to the system of equations (2.32)–(2.34) near the singularity, i.e. as $s \rightarrow -\infty$. The structure of the equations restricts the asymptotic behavior to one of two types. The first type has

$$I: \quad e^\phi = (-s)^\alpha e^{\phi_0} + \dots, \quad e^\rho = (-s)^{-\alpha-1/2} e^{\rho_0} + \dots, \quad \tilde{f} = (-s)^{-2\alpha} f_0 + \dots, \quad (2.40)$$

⁷ By the energy argument of [15], a second mode of the linearized system should also be set to zero in the $z < 2$ case, leading to at best a discrete spectrum of Lifshitz black holes at $z < 2$. This condition is too strong, indicating a problem with the background subtraction used in [15]. An alternative definition of the energy of asymptotically Lifshitz solutions, based on boundary counterterms, recently appeared in [21] for which only a single mode needs to be fine-tuned in order to have finite energy at $z < 2$.

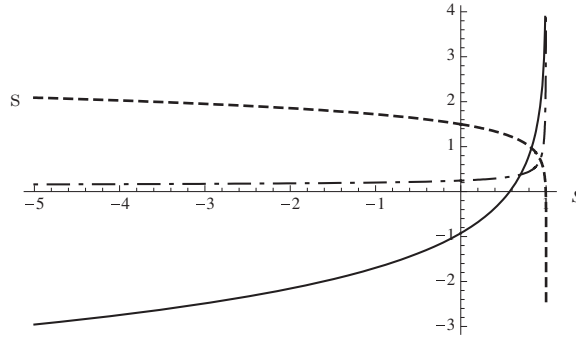


Figure 2. Numerical solution for a $z = 4$ Lifshitz black hole with ρ solid, ϕ dashed and f dot-dashed. The horizon is at $s = 0$ and $r \rightarrow \infty$ corresponds to $s \rightarrow 1$.

where α, ϕ_0, ρ_0 and f_0 are non-universal constants. The assumption that $r \rightarrow 0$ as $s \rightarrow -\infty$ amounts to the requirement $\alpha > 0$ but otherwise these constants are unrestricted *a priori*⁸.

With $\alpha > 0$ the last term in (2.32), the last two terms in (2.33) and the next to last term in (2.34) are sub-leading for this type of solution. The remaining terms in (2.32) cancel automatically at leading order, while (2.33) and (2.34) are satisfied at leading order if

$$\frac{k}{8} e^{2\rho_0+2\phi_0} = -\alpha^2, \quad f_0^2 e^{4\phi_0} = \frac{z}{2}. \tag{2.41}$$

It follows that this type of asymptotic behavior can only be realized in a $k = -1$ geometry. It can be easily checked that the $z = 2$ topological black hole solution is of this type with $\alpha = \frac{1}{2}$.

The other possible behavior near the singularity is

$$\text{II:} \quad e^\phi = (-s)^\beta e^{\phi_0} + \dots, \quad e^\rho = (-s)^{-2\beta-1/2} e^{\rho_0} + \dots, \quad \tilde{f} = f_0 + \dots, \\ \tilde{f}' = (-s)^{-2\beta-1} f_1 + \dots, \tag{2.42}$$

with $\beta > 0$. In this type of solution the last term in (2.32), the fourth and fifth terms in (2.33) and the fourth term in (2.34) are subleading as $s \rightarrow -\infty$. The remainder of (2.32) is automatically satisfied at leading order, while (2.33) and (2.34) require

$$\frac{f_0^2}{8} e^{2\rho_0+4\phi_0} = \beta^2, \quad f_1^2 = \frac{3z}{4} f_0^2 e^{2\rho_0}. \tag{2.43}$$

This time around there is no restriction on the value of k . The exact $z = 4$ black hole solutions fall into this category, with $\beta = \frac{3}{8}$ in the $k = 1$ case and $\beta = \frac{1}{8}$ in the $k = -1$ case.

The Ricci scalar diverges in the $s \rightarrow -\infty$ limit. For type I solutions, the leading behavior is $R = \frac{1}{L^2 r^2} + \dots$, while for type II solutions we find $R = \frac{3f_0^2}{L^2 r^4} + \dots$. The curvature singularity at $r \rightarrow 0$ is always null. This follows immediately from the fact that

$$e^{2\rho} \rightarrow 0, \quad \text{as } s \rightarrow -\infty, \tag{2.44}$$

for both allowed types of asymptotic behavior. As a result, the conformal diagram in figure 1 describes generic asymptotically Lifshitz black holes with a single non-degenerate horizon that are obtained from the action (1.4).

In the notation used in [13], appropriately continued to inside the horizon, a type I solution has $f \propto r^{1-z}, g \propto r^2, h \propto 1$ and $j \propto r^{-1}$, for small r . The corresponding limiting behavior of a solution of type II is $f \propto r^{2-z}, g \propto r^2, h \propto r^{-2}$ and $j \propto r^{-2}$.

⁸ We note that our argument assumes that $r \rightarrow 0$ as $s \rightarrow -\infty$ and does not preclude the existence of solutions with a smooth inner horizon at which $r \rightarrow r_-$ as $s \rightarrow -\infty$ for some finite $r_- > 0$. On the other hand, none of the known exact solutions at $z > 1$ have such an inner horizon.

3. Charged Lifshitz black holes

We wish to extend the notion of a holographic superconductor to systems with Lifshitz scaling. In order to do this, we need to introduce some additional ingredients to the gravity model that we have been considering. As it stands, the model has only a single characteristic length scale and does not support a superconducting transition, or any other phase transition for that matter. If we want to make phase transitions possible, we need to introduce a new scale into the problem. This can be seen explicitly as follows. The gravitational dual of a holographic superconductor is a charged plane-symmetric black hole with hair [4], so for this application we work with $k = 0$ geometries of the form

$$ds^2 = L^2[-e^{2\rho(v,u)} dv du + e^{-2\phi(v,u)}(dx^2 + dy^2)]. \quad (3.1)$$

The plane-symmetric black hole geometry has a scaling symmetry, which is absent for spherical or hyperbolic horizons. The metric is invariant under a global rescaling of the planar coordinates $x, y \rightarrow e^\alpha x, e^\alpha y$ accompanied by a uniform shift $\phi \rightarrow \phi + \alpha$ while keeping ρ, v and u unchanged. On the other hand, such a shift changes the Hawking temperature (2.39). Since different non-zero Hawking temperatures can be mapped into each other by scale transformations they are all physically equivalent in this system and there cannot be any phase transitions. Another way to state this is that in a scale invariant theory at finite temperature there is no other length scale apart from that defined by the temperature itself and therefore different temperatures must be physically equivalent.

There are different ways to add another length scale to the Lifshitz system. We chose to do this by letting our black holes carry an electric charge, which couples to a Maxwell field $\mathcal{F}_{(2)}$. This corresponds to introducing a non-zero charge density into the dual field theoretic system [4] and adds a term to the action

$$S_{\mathcal{F}} = -\frac{1}{2} \int * \mathcal{F}_{(2)} \wedge \mathcal{F}_{(2)}, \quad (3.2)$$

which has the same form as the kinetic term of $F_{(2)}$ and contributes in same way to the field equations. Introducing a new gauge field may, at first sight, appear to be an unnecessary complication given that the model already has a two-form field strength, $F_{(2)}$, and a three-form field strength, $H_{(3)}$, which couples to the two-form field and could be viewed as a form of charged matter. While such an interpretation could in principle be pursued, we find it more useful to consider $F_{(2)}$ and $H_{(3)}$ as auxiliary fields, that only couple to gravity, and whose only role is to give rise to gravitational backgrounds with Lifshitz scaling. We are then free to separately include the physical electromagnetic field $\mathcal{F}_{(2)}$ and introduce charged matter that couples to it but not to the Lifshitz form fields. The manner in which we construct our holographic superconductors with Lifshitz scaling is analogous to the corresponding construction for the $z = 1$ case in [4]. In fact, in the limit $z \rightarrow 1$, our system reduces to the one considered by those authors and we have used this to check our numerical algorithm against known results for $z = 1$.

Writing

$$\mathcal{F}_{(2)} = L p_{vu}(v, u) dv \wedge du, \quad (3.3)$$

the Maxwell equations for $\mathcal{F}_{(2)}$ reduce to

$$\partial_v(e^{-2\rho-2\phi} p_{vu}) = 0 = \partial_u(e^{-2\rho-2\phi} p_{vu}). \quad (3.4)$$

The general solution can be written as

$$p_{vu} = Q e^{2\rho+2\phi}, \quad (3.5)$$

and has a simple interpretation as the Coulomb field of a point charge. Now consider a black hole with charge Q in the s variable. Since the new gauge field does not couple directly to the original Lifshitz gauge fields, the field equation for \tilde{f} (2.32) remains unchanged while equations (2.33) and (2.34) pick up terms involving the black hole charge⁹:

$$0 = s\phi'' + \phi' - 2s\phi'^2 + \frac{k}{4} e^{2\rho+2\phi} + \frac{(z^2 + z + 4)}{8} e^{2\rho} - \frac{1}{4}(\tilde{f}^2 + Q^2) e^{2\rho+4\phi}, \quad (3.6)$$

$$0 = s\rho'' + \rho' - s\phi'^2 + \frac{k}{4} e^{2\rho+2\phi} - \frac{1}{2}(\tilde{f}^2 + Q^2) e^{2\rho+4\phi} - \frac{s}{2z} e^{4\phi} \tilde{f}'^2. \quad (3.7)$$

The field equations can be numerically integrated as before and we will present some numerical results below, but before that we present some exact solutions. In the first example we recover the well-known AdS–Reissner–Nordström solution as a special case with $z = 1$ and $\tilde{f} = 0$. This provides a check of the formalism. The remaining examples are new and describe one-parameter families of charged Lifshitz black holes at $z = 4$.

3.1. AdS–RN black holes at $z = 1$

The AdS–Reissner–Nordström solution at $z = 1$ describes an electrically charged black hole in asymptotically AdS spacetime. In the variables we are using, it is given by

$$e^{2\rho(r_*)} = r^2 + k - \frac{1}{r} \left(r_h^3 + kr_h + \frac{Q^2}{r_h} \right) + \frac{Q^2}{r^2}, \quad (3.8)$$

with $k = +1, 0, -1$ for a spherical, flat or hyperbolic horizon at $r = r_h$. The relationship between the area and tortoise coordinates is

$$\frac{dr}{dr_*} = r^2 + k - \frac{1}{r} \left(r_h^3 + kr_h + \frac{Q^2}{r_h} \right) + \frac{Q^2}{r^2}, \quad (3.9)$$

and the Lifshitz gauge field \tilde{f} is everywhere vanishing. It can be easily checked that

$$e^{2\rho(s)} = \frac{1}{\kappa^2 s} e^{2\rho(r_*)}, \quad e^{-\phi(s)} = r, \quad (3.10)$$

with $s = e^{2\kappa(r_* - r_*^\infty)}$ is a solution of equations (3.6) and (3.7) with $\tilde{f} = 0$.

The Hawking temperature

$$T_H = \frac{1}{4\pi} \left(3r_h + \frac{k}{r_h} - \frac{Q^2}{r_h^3} \right) \quad (3.11)$$

goes to zero as the charge approaches the extremal value for a given horizon area:

$$T_H \rightarrow 0 \quad \text{as} \quad Q^2 \rightarrow Q_{\text{ext}}^2 = 3r_h^4 + kr_h^2, \quad (3.12)$$

as shown in figure 3.

3.2. Exact solutions for charged black holes at $z = 4$

Numerical work is required in order to explore the full parameter range of asymptotically Lifshitz black hole solutions at $z > 1$ but exact solutions are useful, even if they only apply in special cases. We have found a family of exact charged black hole solutions for $z = 4$, which

⁹ For completeness we allow for $k \neq 0$ in the field equations but we are primarily interested in the $k = 0$ case for the application to holographic superconductors.

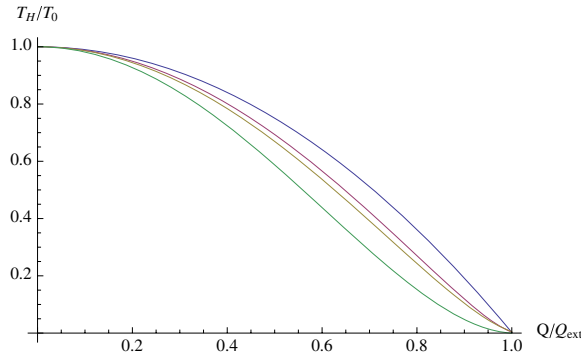


Figure 3. The Hawking temperature of charged black holes with $r_h = 1, k = +1$ for $z = 4, 2, \frac{3}{2}, 1$ from bottom to top. The temperature is normalized to the temperature of the corresponding uncharged black hole while the charge is normalized to the corresponding extremal charge. The curve for $z = 1$ is plotted from the analytic expression (3.11) while the $z > 1$ curves are obtained from numerical solutions.

generalize the isolated $z = 4$ solutions discussed in section 2.1. In the notation of equations (2.2)–(2.3) the metric is given by

$$f(r) = \frac{1}{g(r)} = \sqrt{1 + \frac{k}{10r^2} - \frac{3k^2}{400r^4} - \frac{Q^2}{2r^4}}, \quad (3.13)$$

with $k = +1, 0, -1$. The metric reduces to the $z = 4$ solutions of section 2.1 when $Q = 0$ and $k = \pm 1$ while the $Q = 0, k = 0$ case reduces to the $z = 4$ Lifshitz fixed point geometry.

The tortoise coordinate is given by equation (2.13) with

$$b_1 = \sqrt{\frac{k^2}{100} + \frac{Q^2}{2}} - \frac{k}{20}, \quad (3.14)$$

$$b_2 = \sqrt{\frac{k^2}{100} + \frac{Q^2}{2}} + \frac{k}{20}. \quad (3.15)$$

The change to U, V coordinates in (2.14) leads to a metric of the form (2.15), with the new values of b_1 and b_2 , which is nonsingular at the horizon. The Lifshitz gauge field can be obtained by inserting the $\rho(s)$ and $\phi(s)$ read off from (3.13) into (3.6) and solving for $\tilde{f}(s)$. One finds that $\tilde{f} = \sqrt{6} \left(r^2 + \frac{k}{20} \right)$ is independent of the black hole charge Q .

The Hawking temperature is $T_H = \frac{\kappa}{2\pi}$ with

$$\kappa = Q^2 + \frac{k^2}{50} - \frac{k}{10} \sqrt{\frac{k^2}{100} + \frac{Q^2}{2}}. \quad (3.16)$$

The $k = 0$ metric has $b_1 = b_2$ and takes a particularly simple form in U, V coordinates:

$$ds^2 = L^2 \left(-\frac{r^8}{Q^2} dU dV + r^2 (d\theta^2 + \theta^2 d\varphi^2) \right). \quad (3.17)$$

In this case the horizon is at $r = \frac{Q}{\sqrt{2}}$ and the Hawking temperature is $T_H = \frac{Q^2}{2\pi}$, both of which go to zero in the $Q \rightarrow 0$ limit.

Returning to the more general exact solution in (3.13), the Ricci scalar is given by

$$R = \frac{1}{L^2} \left(\frac{3Q^2}{r^4} + \frac{9k^2}{200r^4} - \frac{3k}{5r^2} - 54 \right) \quad (3.18)$$

and is singular at $r \rightarrow 0$. The curvature singularity is null and the conformal diagram remains the same as in figure 1. In particular, unlike ordinary Reissner–Nordström black holes, these charged Lifshitz black holes do not have an inner horizon away from the singularity at $r \rightarrow 0$. The general analysis from section 2.3 of the asymptotic behavior near the singularity can be extended to the case of charged Lifshitz black holes. They exhibit type II behavior with the replacement $f_0^2 \rightarrow f_0^2 + Q^2$ in equation (2.42).

3.3. Numerical charged black hole solutions

The field equations with the additional Maxwell field included were presented earlier in equations (2.32), (3.6) and (3.7). The initial values giving a smooth horizon at $s = 0$ are modified as follows by a non-vanishing black hole charge:

$$\phi'(0) = \frac{1}{4} \left(-k e^{2\phi(0)} - \frac{(z^2+z+4)}{2} + (\tilde{f}(0)^2 + Q^2) e^{4\phi(0)} \right) e^{2\rho(0)}, \quad (3.19)$$

$$\rho'(0) = \frac{1}{2} \left(-\frac{k}{2} e^{2\phi(0)} + (\tilde{f}(0)^2 + Q^2) e^{4\phi(0)} \right) e^{2\rho(0)} \quad (3.20)$$

$$\tilde{f}'(0) = \frac{z}{2} \tilde{f}(0) e^{2\rho(0)}. \quad (3.21)$$

For any given value of $z > 1$, there is a two-parameter family of asymptotically Lifshitz black hole solutions. If we take the charge Q and $\phi(0)$ as independent parameters then $\tilde{f}(0)$ needs to be fine-tuned in order to obtain the correct asymptotic behavior as $s \rightarrow s_\infty$. Changing the value of $\rho(0)$ amounts to a global rescaling of the s coordinate and does not change the geometry.

The numerical integration proceeds in the same way as before and plots of numerical solutions are qualitatively similar to figure 2. The existence of a new length scale in the system can be seen in the Hawking temperature obtained from (2.39) for black holes with different Q keeping the area coordinate at the horizon fixed. Figure 3 shows the black hole temperature as a function of Q for several values of z . All the black holes have $\phi(0) = 0$. In order to facilitate comparison, the temperature of all the black holes for a given z is normalized to the Hawking temperature of a $Q = 0$ black hole with the same value of z and similarly the charge is normalized to the extremal charge of black holes with the given value of z .

4. Coupling to charged matter

The final ingredient is a scalar field ψ , which is charged under the new gauge field $\mathcal{F}_{(2)}$ but neutral under the original Lifshitz fields $F_{(2)}$ and $H_{(3)}$. This allows our black holes to grow scalar hair and, under certain conditions, the black hole hair corresponds to the condensation of a charged operator at low temperatures in the dual field theory. The overall picture is quite similar to the one obtained for the charged AdS black holes in [4] and extends the notion of a holographic superconductor to systems that exhibit Lifshitz scaling with $z > 1$.

4.1. Charged scalar field

The scalar field action is given by

$$S_\psi = -\frac{1}{2} \int d^4x \sqrt{-g} (g^{\mu\nu} (\partial_\mu \psi^* + iq \mathcal{A}_\mu \psi^*) (\partial_\nu \psi - iq \mathcal{A}_\nu \psi) + m^2 \psi^* \psi), \quad (4.1)$$

where \mathcal{A}_μ are the components of a one-form potential for the two-form field strength $\mathcal{F}_{(2)}$ and q is the charge of ψ . There is an analog in this system of the Breitenlohner–Freedman bound [5] on the allowed mass of the scalar field:

$$L^2 m^2 > -\frac{(z+2)^2}{4}. \tag{4.2}$$

The requirement that the corresponding Euclidean action be finite places restrictions on the asymptotic behavior of the scalar field as $r \rightarrow \infty$. This was worked out in detail for the $z = 2$ case in [12]¹⁰ and, since the analysis carries over to general $z > 1$ in a straightforward way, we merely summarize some results without going into details.

The scalar field equation obtained from (4.1) has two independent solutions $\psi(x^\mu) = c_+ \psi_+(x^\mu) + c_- \psi_-(x^\mu)$ that have the asymptotic form

$$\psi_\pm(x^\mu) \rightarrow r^{-\Delta_\pm} \tilde{\psi}_\pm(\tau, \theta, \varphi) + \dots, \tag{4.3}$$

at large r , where

$$\Delta_\pm = \frac{z+2}{2} \pm \sqrt{\left(\frac{z+2}{2}\right)^2 + m^2 L^2}. \tag{4.4}$$

If the mass squared satisfies

$$L^2 m^2 > 1 - \left(\frac{z+2}{2}\right)^2, \tag{4.5}$$

then only ψ_+ falls off sufficiently rapidly as $r \rightarrow \infty$ and the scalar field has the boundary condition

$$\psi(x^\mu) \rightarrow r^{-\Delta_+} \left(\tilde{\psi}_+(\tau, \theta, \varphi) + O\left(\frac{1}{r^2}\right) \right). \tag{4.6}$$

The scalar field is then dual to an operator of dimension $\Delta_+ > \frac{z}{2} + 2$.

If, on the other hand, the mass squared is in the range

$$1 - \left(\frac{z+2}{2}\right)^2 > L^2 m^2 > -\left(\frac{z+2}{2}\right)^2, \tag{4.7}$$

then both ψ_+ and ψ_- have sufficiently rapid falloff and there is a choice of two different quantizations for the scalar field. In one case the scalar field is asymptotic to ψ_+ and dual to an operator of dimension Δ_+ with $\frac{z}{2} + 1 < \Delta_+ < \frac{z}{2} + 2$. In the other case the scalar field is asymptotic to ψ_- and dual to an operator of dimension Δ_- with $\frac{z}{2} < \Delta_- < \frac{z}{2} + 1$.

In our numerical calculations we set the scalar mass squared to

$$L^2 m^2 = \frac{1}{4} - \left(\frac{z+2}{2}\right)^2 \tag{4.8}$$

which is inside the range where there is a choice of two boundary theories. This value leads to convenient values for the operator dimensions, $\Delta_\pm = \frac{z+2}{2} \pm \frac{1}{2}$. Nonlinear descendants of the leading scalar field modes are suppressed by $O\left(\frac{1}{r^2}\right)$ at $r \rightarrow \infty$ and this choice of mass squared ensures that the first descendant of ψ_- falls off faster than ψ_+ .

¹⁰ The discussion in [12] was in turn based on earlier work on symmetry breaking in the context of the AdS/CFT correspondence [22].

4.2. Lifshitz black holes with scalar hair

We will look for static spherically symmetric black hole solutions using a metric of the form (2.18). We make the gauge choice

$$\mathcal{A} = L(a_v dv + a_u du), \quad (4.9)$$

with $a_v = a_u \equiv a$. In this gauge, the Maxwell equations for static configurations imply the equation

$$\psi^* \frac{d\psi}{dr_*} - \psi \frac{d\psi^*}{dr_*} = 0, \quad (4.10)$$

which implies, in turn, that the phase of ψ is constant and we can take the scalar field to be real valued. The remaining non-trivial Maxwell equation is

$$\frac{d}{dr_*} \left(e^{-2\rho-2\phi} \frac{da}{dr_*} \right) + q^2 L^2 \psi^2 e^{-2\phi} a = 0, \quad (4.11)$$

which agrees with (3.4) if $q = 0$, i.e. when ψ is a neutral scalar field that does not act as a source for the Maxwell field.

The next step is to write equations for static configurations using the s variable, in which the metric is non-degenerate at the horizon. Since a is a component of a one-form potential, it transforms under a change of coordinates:

$$a(r_*) = \frac{ds}{dr_*} a(s) = 2\kappa s a(s). \quad (4.12)$$

It follows that the potential vanishes at the horizon in tortoise coordinates as long as it remains a smooth function there in the s variable. The Maxwell equation becomes

$$\frac{d}{ds} \left(e^{-2\rho-2\phi} \left(a + s \frac{da}{ds} \right) \right) + \frac{q^2 L^2}{4} \psi^2 e^{-2\phi} a = 0, \quad (4.13)$$

and for a smooth horizon at $s = 0$ the initial data must satisfy

$$-a'(0) + a(0)(\rho'(0) + \phi'(0)) = \frac{q^2 L^2}{8} \psi(0)^2 a(0) e^{2\rho(0)}. \quad (4.14)$$

The value of $a(0)$ is a free parameter that determines the black hole charge. To see this, we note that for $q = 0$ we have $a + sa' = \frac{Q}{4} e^{2\phi+2\rho}$, with Q the black hole charge, and in this case

$$a(0) = \frac{Q}{4} e^{2\phi(0)+2\rho(0)}. \quad (4.15)$$

If there is a non-vanishing scalar field with $q \neq 0$, then the solution of the Maxwell equation will no longer be a simple Coulomb field but we are nevertheless free to write $a(0)$ the same way as in (4.15) and the constant Q can still be interpreted as the total charge inside the black hole.

The scalar field equation in the s variable is

$$0 = s\psi'' + \psi' - 2s\phi'\psi' + 4sq^2L^2a^2\psi - \frac{m^2L^2}{4} e^{2\rho}\psi, \quad (4.16)$$

with the following condition on initial data at a smooth horizon:

$$\psi'(0) = \frac{m^2L^2}{4} e^{2\rho(0)}\psi(0). \quad (4.17)$$

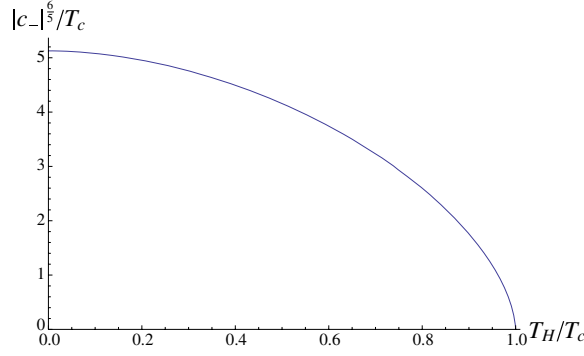


Figure 4. Holographic superconductivity at $z = 3/2$. The graph shows the condensate in the \mathcal{O}_- theory as a function of temperature normalized to the critical temperature.

The equation for \tilde{f} remains unchanged but the remaining two field equations get contributions from the scalar action:

$$0 = s\phi'' + \phi' - 2s\phi'^2 + \frac{k}{4} e^{2\rho+2\phi} + \frac{(z^2+z+4)}{8} e^{2\rho} - \frac{1}{4} \tilde{f}^2 e^{2\rho+4\phi} - 4(a + s a')^2 e^{-2\rho} - \frac{m^2 L^2}{16} e^{2\rho} \psi^2, \quad (4.18)$$

$$0 = s\rho'' + \rho' - s\phi'^2 + \frac{k}{4} e^{2\rho+2\phi} - \frac{1}{2} \tilde{f}^2 e^{2\rho+4\phi} - \frac{s}{2z} e^{4\phi} \tilde{f}'^2 - 8(a + s a')^2 e^{-2\rho} + \frac{s}{4} \psi'^2 - sq^2 L^2 a^2 \psi^2, \quad (4.19)$$

and the conditions on the initial data become

$$\phi'(0) = \frac{1}{4} \left(-k e^{2\phi(0)} - \frac{(z^2+z+4)}{2} + (\tilde{f}(0)^2 + Q^2) e^{4\phi(0)} + \frac{m^2 L^2}{4} \psi(0)^2 \right) e^{2\rho(0)}, \quad (4.20)$$

$$\rho'(0) = \frac{1}{2} \left(-\frac{k}{2} e^{2\phi(0)} + (\tilde{f}(0)^2 + Q^2) e^{4\phi(0)} \right) e^{2\rho(0)}. \quad (4.21)$$

For given values of z , k , q and m^2 there is now a three-parameter family of smooth initial values for the dynamical fields at the horizon given by Q , $\psi(0)$ and $f(0)$. As before, the initial data must be fine-tuned to obtain an asymptotically Lifshitz geometry and we take the black hole charge Q and the value of the scalar field at the horizon $\psi(0)$ as independent parameters.

4.3. Superconducting phase

In order to study black hole geometries that are dual to a holographic superconductor we set $k = 0$. For a given value of z we set the scalar mass squared to the value in (4.8) and select a value for the scalar field charge q . We then find numerical black hole solutions with scalar hair for a range of Q and $\psi(0)$ and study the asymptotic large r behavior of the scalar hair in each case. The signal of a superconducting condensate in one or the other boundary theory is to have either

$$c_+ = 0 \quad \text{and} \quad \langle \mathcal{O}_- \rangle = c_- \neq 0 \quad (4.22)$$

or

$$c_- = 0 \quad \text{and} \quad \langle \mathcal{O}_+ \rangle = c_+ \neq 0. \quad (4.23)$$

The curve of vanishing c_+ (c_-) in the Q versus $\psi(0)$ plane can be tracked, tabulating the value of c_- (c_+) as a function of the black hole temperature along the way. Figure 4 shows the result of this procedure for hairy Lifshitz black holes at $z = 3/2$ for the scalar field charge $q = 1$. There is clear evidence of condensation in the corresponding boundary theory. Analogous results are obtained for other values of q and more general $z > 1$ but we defer a more detailed study of the parameter space to [23].

5. Conclusions

In this paper we have given a holographic description of superconductors with Lifshitz scaling. The key ingredient is a family of black holes with scalar hair in an asymptotically Lifshitz space time. These black holes are interesting in their own right, and we have studied their geometry in detail uncovering several intriguing properties, including the presence of null singularities. Our work is an extension of the previous work on Lifshitz black holes in [13–15].

As in the case of the more conventional holographic superconductors at $z = 1$, see [1, 2], our black holes need to be equipped with additional charges in order for a phase transition to a superconducting phase to occur. We provide such a construction and show how a charged scalar in this background condenses at a critical temperature, indicating a superconducting phase.

Our results demonstrate that phase transitions leading to superconductivity at low temperature, can be given a holographic description also for Lifshitz points. Such superconductors are known from solid state physics, see e.g. [24], and we hope that our results will be useful for further studies of these systems. Some of their properties are different from usual superconductors, and it would be interesting to see whether this has a holographic counterpart.

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References

- [1] Gubser S S 2008 Breaking an Abelian gauge symmetry near a black hole horizon *Phys. Rev. D* **78** 065034 (arXiv:0801.2977 [hep-th])
- [2] Hartnoll S A, Herzog C P and Horowitz G T 2008 Building a holographic superconductor *Phys. Rev. Lett.* **101** 031601 (arXiv:0803.3295 [hep-th])
- [3] Maldacena J M 1998 The large N limit of superconformal field theories and supergravity *Adv. Theor. Math. Phys.* **2** 231
Maldacena J M 1999 *Int. J. Theor. Phys.* **38** 1113 (arXiv:hep-th/9711200)
- [4] Hartnoll S A, Herzog C P and Horowitz G T 2008 Holographic superconductors *J. High Energy Phys.* **JHEP0812(2008)015** (arXiv:0810.1563 [hep-th])
- [5] Breitenlohner P and Freedman D Z 1982 Stability in Gauged extended supergravity *Ann. Phys.* **144** 249
- [6] Hartnoll S A 2009 Lectures on holographic methods for condensed matter physics *Class. Quantum Grav.* **26** 224002 (arXiv:0903.3246 [hep-th])
- [7] Herzog C P 2009 Lectures on holographic superfluidity and superconductivity *J. Phys. A: Math. Theor.* **42** 343001 (arXiv:0904.1975 [hep-th])

- [8] Rokhsar D S and Kivelson S A 1988 Superconductivity and the quantum hard-core dimer gas *Phys. Rev. Lett.* **61** 2376
- [9] Ardonne E, Fendley P and Fradkin E 2004 Topological order and conformal quantum critical points *Ann. Phys.* **310** 493 (arXiv:cond-mat/0311466)
- [10] Fradkin E, Huse D A, Moessner R, Oganesyan V and Sondhi S L 2004 Bipartite Rokhsar–Kivelson points and Cantor deconfinement *Phys. Rev. B* **69** 224415 (arXiv:cond-mat/0311353)
- [11] Vishwanath A, Balents L and Senthil T 2004 Quantum criticality and deconfinement in phase transitions between valence bond solids *Phys. Rev. B* **69** 224416 (arXiv:cond-mat/0311085)
- [12] Kachru S, Liu X and Mulligan M 2008 Gravity duals of Lifshitz-like fixed points *Phys. Rev. D* **78** 106005 (arXiv:0808.1725 [hep-th])
- [13] Danielsson U H and Thorlacius L 2009 Black holes in asymptotically Lifshitz spacetime *J. High Energy Phys.* **JHEP0903(2009)070** (arXiv:0812.5088 [hep-th])
- [14] Mann R B 2009 Lifshitz topological black holes *J. High Energy Phys.* **JHEP0906(2009)075** (arXiv:0905.1136 [hep-th])
- [15] Bertoldi G, Burrington B A and Peet A 2009 Black holes in asymptotically Lifshitz spacetimes with arbitrary critical exponent *Phys. Rev. D* **80** 126003 (arXiv:0905.3183 [hep-th])
- [16] Bertoldi G, Burrington B A and Peet A W 2009 Thermodynamics of black branes in asymptotically Lifshitz spacetimes *Phys. Rev. D* **80** 126004 (arXiv:0907.4755 [hep-th])
- [17] Taylor M 2008 Non-relativistic holography arXiv:0812.0530 [hep-th]
- [18] Azeanagi T, Li W and Takayanagi T 2009 On string theory duals of Lifshitz-like fixed points *J. High Energy Phys.* **JHEP0906(2009)084** [arXiv:0905.0688 [hep-th]]
- [19] Pang D W 2009 A note on black holes in asymptotically Lifshitz spacetime arXiv:0905.2678 [hep-th]
- [20] Birmir B, Giddings S B, Harvey J A and Strominger A 1992 Quantum black holes *Phys. Rev. D* **46** 638 (arXiv:hep-th/9203042)
- [21] Ross S F and Saremi O 2009 Holographic stress tensor for non-relativistic theories *J. High Energy Phys.* **JHEP0909(2009)009** (arXiv:0907.1846 [hep-th])
- [22] Klebanov I R and Witten E 1999 AdS/CFT correspondence and symmetry breaking *Nucl. Phys. B* **556** 89 (arXiv:hep-th/9905104)
- [23] Brynjolfsson E J, Danielsson U H, Thorlacius L and Zingg T 2009 in preparation
- [24] Buzdin A I and Kulic M L 1984 Unusual behaviour of superconductors near the tricritical Lifshitz point *J. Low Temp. Phys.* **54** 203